CHAPTER 4
FUZZY LOGIC

❖ Classical Logic

Terms:

❖ Logic: The study of the methods and principles of reasoning in all its possible forms.

❖ Proposition:

Ex: "it rained on Tuesday"

❖ Logic variable:

Propositions or statements about the world that may be either true or false.

❖ One area of logic, propositional logic (calculus), deal with combinations of logic variables that stand for arbitrary propositions.

Example:

\[ P \land Q \]

\[(P \land Q) \lor \neg P \Rightarrow R \]

where \( P, Q \) and \( R \): logic variables,

\( \land, \lor, \neg, \Rightarrow \) : logic connectives

So, it is a main concern to study the rules by which new logic variables (new hypothetical proposition) can be produced as function of some given logic variables.

**Def:** A logic function is a function that assigns a particular truth value to a new variable for each combination of truth values of given variables.

**Example:**

\[
\begin{array}{c|c|c|c|c|c|c}
 P & Q & \neg P & \neg P \lor Q & P \Rightarrow Q & \neg (P \lor Q) = (P \Rightarrow Q) \\
 T & T & F & T & T & T \\
 T & F & F & F & F & T \\
 F & T & T & T & T & T \\
 F & F & T & T & T & T \\
\end{array}
\]

**Def:** Logic functions of one or two variables are usually called logic operations.

**Example:** Logic functions of two variables
**Def:** The basis of the logic function, which is constructed with some logic operations, is called **logic primitive.**

A set of logic primitives is **complete** iff (if and only if) any logic function of variables $v_1, v_2, \ldots, v_n$ (for any finite $n$) can be composed by a finite number of these primitives.

**Example:**
Two important examples of many complete sets of primitives:

1. *negation* ($\neg$), *conjunction* ($\wedge$), *disjunction* ($\vee$)
2. *negation* and *implication* ($\Rightarrow$)

**Def:** The **logic formulas** are algebraic expressions defined recursively as follows:

1. The truth values 0 and 1 are logic formula.
2. If $v$ denotes a logic variable, then $v$ and $\neg v$ are logic formula.
3. If $a$ and $b$ denote logic formulas, then $a \wedge b$ and $a \vee b$ are also logic formulas.
4. The only logic formulas are those defined by statements 1 through 3.

**Remark:**

1. Every logic formula of this type defines a logic function by composing it from the three primary functions.
2. The order in which the individual compositions are to be performed must be specified to define a unique function.
3. Other types of logic formulas can be defined by replacing, for example, $\wedge$ and $\vee$ with $\Rightarrow$ in the definition. (i.e. negation and implication)

**Example:**

- **logic connectives:** $\neg, \wedge$
- **logic variables:** $v_1, v_2$
- **logic primitives:** $\neg, \wedge$

\[
\left( \neg v_1 \wedge \neg v_2 \right)
\]

- **logic operations:** $\neg v_1, \neg v_2, v_1 \wedge v_2$
- **logic functions:** $\neg v_1, \neg v_2, v_1 \wedge v_2$
- **logic formulas:** $\neg v_1, \neg v_2, v_1 \wedge v_2$

**Def:**

1. When the variables represented by a logic formula is always truth, it is called a **tautology.**
2. When it is always false, it is called a **contradiction.**

**Example:**

Refer to Table, when $a = b$,

"$a \leftrightarrow b$: biconditional" a tautology

"$a \oplus b$: exclusive-or" a contradiction
Remark:

1. **Tautologies** are important for *deductive reasoning* since they represent logic formula that are always true.

2. Various forms of tautologies can be used for making deductive inference, called *inference rules*.

   Ex: Some tautologies frequently used as inference rules:
   1. \((a \land (a \Rightarrow b)) \Rightarrow b\) (modus ponens)
   2. \((\bar{b} \land (a \Rightarrow b)) \Rightarrow \bar{a}\) (modus tollens)
   3. \(((a \Rightarrow b) \Rightarrow (b \Rightarrow c)) \Rightarrow (a \Rightarrow c)\) (hypothetical syllogism)

3. Every tautology remains a tautology when any of its variables is replaced with any arbitrary logic formula.

   Ex:
   \((a \land (a \Rightarrow b)) \Rightarrow b\) (a tautology)
   \(\downarrow\)
   \(((a \lor b) \land ((a \lor b) \Rightarrow \bar{b})) \Rightarrow \bar{b}\) (a tautology)

Comment:

1. It is well established that *propositional logic*, *set theory* and *Boolean algebra* are mutual *isomorphic* under appropriate correspondence between components of these three mathematical systems.

2. The *isomorphisms* between these three mathematical systems guarantee that every theorem in any one of these theories has a counterpart in each of the other two theories.

3. Our study in the operation on fuzzy set can be applied to the inference in fuzzy logic.

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**Propositions** are sentences expressed in some language. A simple proposition (logic variable) can be expressed, in general, as:

\[ x \text{ is } P \]

where

- \(x\): a symbol of a *subject*;
- \(P\): a *predicate*, which characterizes a *property*.

Example:

"Austria is a German-speaking country"
\[ \downarrow \downarrow \]
subject predicate

Note:

The primary focus of fuzzy logic is on natural language, where approximate reasoning with imprecise proposition is rather typical.
Example:
Old coins are usually rare collectibles.
Rare collectibles are expensive.

Old coins are usually expensive.

\[(a \land (a \Rightarrow b)) \Rightarrow b\] (modus ponens)

Comment:
1. Since new proposition (predicates, logic variables) are to be produced from given propositions by logic formulas, we need mathematic concepts to evaluate the logic functions defined by the logic formulas.

2. A basic principle that allows the generalization of crisp mathematical concepts to the fuzzy framework is known as the extension principle.

3. 

\[\begin{array}{c}
X \\
\uparrow \quad f \\
\hline \\
Y \\
\end{array} \quad f^{-1} \quad \begin{array}{c}
\tilde{X} \\
\downarrow \\
\hline \\
\tilde{Y} \\
\end{array}
\]

extension principle

Def:
a) Given \( f: X \rightarrow Y \) and any fuzzy set \( A \in \tilde{P}(X) \), where
\[
A = \mu_1/x_1 + \mu_2/x_2 + \cdots + \mu_n/x_n
\]
the extension principle states that
\[
f(A) = f(\mu_1/x_1 + \mu_2/x_2 + \cdots + \mu_n/x_n)
= \mu_1/f(x_1) + \mu_2/f(x_2) + \cdots + \mu_n/f(x_n)
\]

If more than one element of \( X \) is mapped by \( f \) to the same element \( y \in Y \), then the maximum of the membership grades of these elements in the fuzzy set \( A \) is chosen as the membership grade for \( y \) in \( f(A) \).

b) Let \( X = X_1 \times X_2 \times \cdots \times X_r \), and \( \tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_r \) be \( r \) fuzzy sets in \( X_1, X_2, \ldots, X_r \), respectively. \( f: X \rightarrow Y, \ y = f(x_1, x_2, \ldots, x_r) \). Then a fuzzy set \( \tilde{B} \) in \( Y \) is defined by
\[
\tilde{B} = \{ (y, \mu_{\tilde{B}}(y)) | y = f(x_1, \ldots, x_r), (x_1, \ldots, x_r) \in X \}
\]
\[
\mu_{\tilde{B}}(y) = \left\{ \begin{array}{ll}
\sup_{(x_1, \ldots, x_r) \in f^{-1}(y)} \min \{\mu_{\tilde{A}_1}(x_1), \ldots, \mu_{\tilde{A}_r}(x_r)\} & \text{if } f^{-1}(y) \neq \emptyset \\
0 & \text{otherwise}
\end{array} \right.
\]

where \( f^{-1} \) is the inverse of \( f \).

For \( r = 1 \) the extension principle reduces to
\[
f(\tilde{A}) = f \{ (x, \mu(x)) | x \in X \}
\]
\[
\mu_{\tilde{A}}(y) = \left\{ \begin{array}{ll}
\sup_{x \in f^{-1}(y)} \mu_{\tilde{A}}(x) & \text{if } f^{-1}(y) \neq \emptyset \\
0 & \text{otherwise}
\end{array} \right.
\]

This is the same as a).
Remark:
The extension principle can and has been modified by using probability sum rather than sup and the probability product rather than min.

Example:
Let \( X = \{x_1, x_2, \ldots, x_7\} = \{1, 2, 3, 4, 5, 6, 7\} = 1/1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 \)
Let \( \tilde{A} = \text{"small"} = 1/1 + 1/2 + .8/3 + .5/4 \)
If \( f(x) = x^2 \), then \( f(\text{small}) = \)

Example:
Let
\[
\tilde{A} = \{(-1,.5), (0,.8), (1,1), (2,.4)\} = .5/1 + .8/0 + 1/1 + .4/2
\]
If \( f(x) = x^2 \), then \( f(\tilde{A}) = \)

Example:
Let 
\[
X_1 = X_2 = 1 + 2 + 3 + 4 + 5 + 6 + 7
\]
\( \tilde{A}_1 = \text{approximately} 2 = .6/1 + 1/2 + .8/3 \)
\( \tilde{A}_2 = \text{approximately} 4 = .8/3 + 1/4 + .7/5 \)
If * denotes the arithmetic product of \( \tilde{A}_1 \) and \( \tilde{A}_2 \), then
\[
\tilde{A}_1 \ast \tilde{A}_2 =
\]

Example:
Let \( f \) be specified by the following matrix:
\[
\begin{bmatrix}
x & y \\
a & p & p \\
b & q & r \\
c & r & p \\
\end{bmatrix}
\]
Let \( \tilde{A}_1 \) be a fuzzy set defined on \( X_1 \) and let \( \tilde{A}_2 \) be a fuzzy set defined on \( X_2 \) such that
\[
\tilde{A}_1 = .3/a + .9/b + .5/c
\]
\[
\tilde{A}_2 = .5/x + 1/y
\]
The membership grades of \( p, q, \) and \( r \) in fuzzy set
\[
B = f(A_1, A_2) \in \tilde{P}(Y) =
\]
Comment: Recall the syllogism example:

Old coins are usually rare collectibles.
Rare collectibles are expensive.

Old coins are usually expensive.

- One of the basic tools for fuzzy logic and approximate reasonings is the notion of a linguistics variables.

- In order to define linguistic variables and deal with inference of natural language as above, fuzzy logic allows the use of:

1. fuzzy predicates: expensive, old, young;
2. fuzzy quantifiers: many, few;
3. fuzzy truth value: truth false;
4. fuzzy modifiers (hedges): very, more or less.

Example:

Let $X$ be a linguistic variable with label "Age" with the age in years within $U=[0,100]$.

The term set could be as $T(Age)=\{\text{old, very old, quite young, more or less young,} \ldots\}$.

$G(x)$ is a syntactic rule which generates the (labels of) terms in the term set.

$\tilde{M}(X)$ is the rule that assigns a meaning (a fuzzy subset) to the terms.

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$\tilde{M}(old) = \{(u, \mu_{\text{old}}(u)) | u \in [0,100]\}$

$\mu_{\text{old}}(u) = \begin{cases} 0 & u \in [0,50] \\ 1 + \left(\frac{u-50}{5}\right)^{-1} & u \in [50,100] \end{cases}$

Def: A linguistic variable is characterized by a quintuple $(x, T(x), U, G, \tilde{M})$ in which

- $x$ the name of the variable;
- $T(x)$ the term-set of $x$, that is, the set of names of linguistic values of $x$;
- $U$ the universal discourse in which the linguistic variable defined;
- $G$ a syntactic rule for generating the name of $x$;
- $\tilde{M}$ a semantic rule for associating with each $X$ its meaning, $\tilde{M}(X)$, which is a fuzzy subset of $U$. 

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1. In the generating of the term-set of the linguistic variable, an operation called **linguistic modifier** (*linguistic hedge*) is applied to modify the meaning of a term or more general of a fuzzy set.

2. Mathematical models frequently used for modifiers are:

   **Concentration:**
   \[ \mu_{\text{Con}(A)}(u) = (\mu_A(u))^2 \]

   **Dilation:**
   \[ \mu_{\text{Dil}(A)}(u) = (\mu_A(u))^{3/2} \]

   **Contrast intensification:**
   \[ \mu_{\text{Int}(A)}(u) = \begin{cases} 
   2(\mu_A(u))^2 & \text{for } \mu_A(u) \in [0,0.5] \\
   1 - 2(1 - \mu_A(u))^2 & \text{otherwise} 
   \end{cases} \]

**Example:** Let

\[ \mu_{\text{old}}(u) = \begin{cases} 
   0 & u \in [0,50] \\
   \left(1 + \frac{u - 50}{5}\right)^{-1} & u \in [50,100] 
   \end{cases} \]

When \( u = 60 \), \( \mu_{\text{old}}(60) = 0.8 \); \( \mu_{\text{very old}}(60) = \) \( \mu_{\text{fairly old}}(60) = \)

Let's consider the following four fuzzy predicates:

- Tina is young is true.
- Tina is young is false.
- Tina is young is fairly true.
- Tina is young is very false.
Remark:

The truth value of the proposition depends not only on the membership grade of Tina's age in the fuzzy set but also depends upon the strength of truth claimed (the hedge).

Example:

Let the universal discourse be $X = \{1, 2, 3, 4\}$. Fuzzy set $\widetilde{A} = \text{little} = \{(1, 1), (2, 0.6), (3, 2), (4, 0)\}$

Let

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x$ is little and $y$ are approximately equal</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.5</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0</td>
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<tr>
<td>2</td>
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<tr>
<td>2</td>
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<td>2</td>
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<tr>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0.5</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

⇒

For $y = 1$:

For $y = 2$:

For $y = 3$:

For $y = 4$:
Remark:

In fact the table

\[
\begin{array}{cccc}
1 & 1 & 1 \\
1 & 2 & 0.5 \\
1 & 3 & 0 \\
1 & 4 & 0 \\
2 & 2 & 1 \\
2 & 3 & 0.5 \\
2 & 4 & 0 \\
3 & 3 & 1 \\
3 & 4 & 0.5 \\
4 & 4 & 1 \\
\end{array}
\]

is formally called a fuzzy relation defined in \( X \times X \). Rewrite the table as \( \tilde{R} \):

\[
\begin{array}{cccc}
1 & 1 & .5 & 0 & 0 \\
.5 & 1 & .5 & 0 \\
0 & .5 & 1 & .5 \\
0 & 0 & .5 & 1 \\
\end{array}
\]

By applying the extension principle, the reasoning can be deduced by solving the composing equation

\[
\tilde{B}(y) = \tilde{A}(x) \circ \tilde{R}(x, y)
\]

where

\[
\tilde{R}(y) = \max_x \min \{ \mu_A(x), \mu_B(x, y) \}
\]

Example:

Comment:

1. To study the fuzzy relation is a main course in fuzzy set theory.

2. How to build an expert system by applying the fuzzy propositional logic and fuzzy approximately reasoning.